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Marangoni convection

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Abstract : Aspects of Marangoni convection induced by heating from below are reviewed. The effect of rotation is discussed with the emphasis on the oscillatory modes of the system which can be released in a Hopf bifurcation.

Keywords : Convection, rotation, Hopf bifurcation

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1. Introduction

After the rain water has evaporated from hot dusty road, what is left behind is an almost regular patterns of hexagons traced out by the dust. This is a fairly common observation when the first rains arrive in most of the tropical countries. The physical phenomenon underlying this observation is convection [1,2]. Generally one associates convection with buoyancy – the hot fluid below rises upwards and the cold fluid at the top tumbles down when a container of liquid is heated from below. The motion takes place with periodicity in the horizontal plane in the absence of any imperfections. If the fluid layer is very thin as happens when the rain starts falling on the hot road surface, the driving force which causes convection is not buoyancy but surface tension. The surface tension driven convection is called Marangoni convection. At the free surface of the fluid, surface tension forces come into play and for a sufficiently thin fluid layer, the surface forces can cause convective motion in the bulk. If one carries out this experiment in a controlled environment, then a quantitative measure can be had of when the convection begins. The temperature gradient in the fluid has to be large enough before a convective motion can be sustained. For sufficiently small gradients, any attempted movement will be damped out by viscosity and thermal conductivity. The gradient has to exceed a critical value before convection begins. Assuming ideal conditions, the convective pattern will be periodic in the horizontal plane. What would be the wavelength of the periodicity ? That too can be answered from controlled experiments. In a laboratory one takes a fluid layer of height 'd' in a metallic container of lateral extension L and heats from the bottom plate. At the surface of the fluid there is a liquid-air interface. In general $L >> d$ and the system acts like an infinite system, with wall effects completely absent. The heating of the bottom surface is controlled

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and a temperature gradient β exists in the liquid layer. This gradient is known in terms of the bottom plate temperature T_1 (a conducting material for the plate ensures that the fluid layer in contact with the plate shares the plate temperature), the ambient temperature T_0 and the rate of radiation H from the topsurface. In the absence of any convection, $\nabla^2 T = 0$, which for a laterally infinite system reduces to $\frac{d^2T}{dz^2} = 0$ (translational invariance in x and y directions imply that T is a function of z only). The solution is $T = T_1 + \beta z$, where β is the gradient. The gradient at the surface is proportional to the temperature difference with the surroundings and is thus $-H(T_{Surface} - T_0) = -h(T_1 + \beta d - T_0)$, where H is a positive constant. Since this must equal the gradient β in the fluid, we have

$$\beta = -\frac{H(T_1 - T_0)}{1 + dH} = -\frac{H\Delta T}{1 + dH}. \tag{1}$$

For a given value of the gradient β, the state where the heat transport is by conduction alone may not be a stable state. The relevant dimensionless quantity proportional to β which determines the stability and acts as a control parameter is the Marangoni number M which is defined as

$$M = \frac{\alpha\beta d^3 S_0}{\lambda\rho\nu} \tag{2}$$

where $\alpha = \frac{1}{S_0}\frac{\partial S}{\partial T}$ is a parameter which shows how sensitive surface tension is to temperature, S_0 is the surface tension at the reference temperature T_0, ρ is the density, ν is the kinematic viscosity and λ is the thermal diffusivity. For small M, the conduction state is stable. One of the tasks of theoretical considerations is to determine the critical value M_c of M at which the conduction state looses stability [3, 4].

The laboratory experiment of heating the fluid with a free surface from below can be made more interesting if the container holding the fluid can be made to rotate about the vertical axis with a speed Ω. A rotating layer of fluid confined at the bottom and free at the top is like the atmospheric boundary layer and should be supporting the same sort of wave motions as those occurring in atmospheric flows. In the laboratory fluid, these waves would be damped by viscosity and thermal conductivity. An interesting issue is whether the Marangoni heating would be able to overcome the dissipation and reflect the presence of the waves in the onset of oscillatory convection. In this article, we will discuss the onset of Marangoni convection in a rotating fluid.

2. Models and boundary conditions

The basic governing the hydrodynamic flow are

i) Navier Stokes' equation and

ii) heat conduction equation, which can be written as

$$\frac{\partial \boldsymbol{V}}{\partial t} + (\boldsymbol{V}\cdot\nabla)\boldsymbol{V} = \nu\nabla^2 \boldsymbol{V} - \frac{\nabla p'}{\rho} + 2(\boldsymbol{\Omega}\times\boldsymbol{V}) \tag{3}$$

$$\frac{\partial T}{\partial t} + (\boldsymbol{V}\cdot\nabla)\,T = \lambda\nabla^2 T, \tag{4}$$

where Ω is the rotation speed, p' is the pressure corrected for the centrifugal force and ρ is the density. There is no gravity for the steady conduction state, $V = 0$ and the hydrostatic pressure p' is a constant. The temperature profile is $T_s = T_1 - \beta_z$, where β is given by eq. (1). We need to carry out a linear stability analysis of the state. To do so we write the velocity and temperature fields as $\boldsymbol{v} = 0 + \boldsymbol{u}$, and $T = T_s + \delta T$, insert these forms in eqs. (3) and (4) and linearize in $\boldsymbol{u}$ and δT. All quantities we made dimensionless by appropriate scaling. The dimensionless z-component of $\boldsymbol{u}$ is w and the dimensionless form of δT is θ. For a laterally infinite system, the fields will be periodic with wavenumber $\boldsymbol{a}$ in the $x - y$ plane. Accordingly

$$w = e^{pt} W(z) e^{ia_1 x + ia_2 y}, \tag{5}$$

$$\theta = e^{pt} \Theta(z) e^{ia_1 x + ia_2 y}, \tag{6}$$

where a_1 and a_2 are components of 'a' and e^{pt} expresses the time dependence. We note that vorticity $\boldsymbol{\omega}$ is defined as $\boldsymbol{\omega} = \nabla \times \boldsymbol{v}$, its z-component will be denoted by ω_z, which we take to be of the form

$$\omega_z = e^{pt} Z(z) e^{ia_1 x + ia_2 y}. \tag{7}$$

Two curls of eq. (3) supplemented by the continuity equation leads to

$$\nabla^2\left(\nabla^2 - \frac{\partial}{\partial t}\right)w = \tau D\omega_z \tag{8}$$

where $\tau = \dfrac{2\Omega d^2}{\nu}$, $D = \dfrac{\partial}{\partial z}$ and $T = \tau^2$ is the Taylor number. The z-component of the curl of eq. (3) gives

$$\left(\nabla^2 - \frac{\partial}{\partial t}\right)w_z = -\tau Dw. \tag{9}$$

The linear stability equation following from eq. (4) is

$$\left(\sigma\frac{\partial}{\partial t} - \nabla^2\right)\theta = w, \tag{10}$$

where $\sigma = \frac{\nu}{\lambda}$ is the Prandlt number.

The linear stability problem is posed by eqs. (8), (9) and (10). With the sort of solution used in eqs. (5)-(7), we have

$$(D^2 - a^2)(D^2 - a^2 - p)W = \tau DZ, \tag{11}$$

$$(D^2 - a^2 - p)Z = \tau DW, \tag{12}$$

$$(D^2 - a^2 - \sigma p)\Theta = -W. \tag{13}$$

The task is to find under what condition does the solution of eqs. (11)-(13) lead to an eigenvalue of p with the non negative real part. However, before we can proceed, we need to specify the boundary conditions. The three equations require 8 boundary conditions, 4 on W, 2 on Z and 2 on Θ. On the rigid thermally conducting boundary at $z = 0$, $W = DW = Z = \Theta = 0$. On the free surface, $z = 1$, we have $DZ = 0$. The free surface allows for fluctuations of the surface and if η is the surface deformation at any point, then $w = \frac{d\eta}{dt}$. Expressing the deformation as $\eta = Ae^{pt}e^{i(a_1x+a_2y)}$, $W = pA$ on $z = 1$. The heat flux at $z = 1$ leads to $D\Theta = \Gamma(A + \Theta)$. We now turn to the force balance on the surface $z = 1$. The forces on the surface are obtainable from the stress tensor T_{ij} and the surface tension. The stress tensor is given by

$$T_{ij} = p'\delta_{ij} + \rho\nu\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right). \tag{14}$$

In the direction along the normal, the change in stress tensor equals the surface tension times the sum of the inverse of the principle radii of curvature and leads to .

$$-\frac{d^2}{\rho\nu\lambda}\delta p' + \frac{gd^2}{\nu\lambda}\eta + 2\frac{\partial w}{\partial z} = \frac{1}{Cr}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\eta, \tag{15}$$

where $Cr = \frac{\rho\nu\lambda}{2Sd}$ is the Crispation number and $\delta p'$ is the pressure fluctuation. For an incompressible flow $\nabla^2\beta p' = 0$. A derivative with respective to z of the z-component of the linearized fluctuation of eq. (3) yields

$$\left(\frac{\partial}{\partial t} - \nabla^2\right)\frac{\partial w}{\partial z} = -\frac{\partial^2}{\partial z^2}\frac{\delta p'}{\rho} = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\frac{\delta p'}{\rho}. \tag{16}$$

Operating on eq. (15) with $\nabla_H^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$,

$$-\left(\frac{\partial}{\partial t} - \nabla^2\right)\frac{\partial w}{\partial z} + 2\nabla_H^2\frac{\partial w}{\partial z} + \frac{gd^3}{\nu\lambda}\nabla_H^2\eta = \frac{1}{Cr}\nabla_H^4\eta. \tag{17}$$

Using the form of w and η as discussed before

$$Cr(-p + D^2 - 3a^2)DW = a^2(B + a^2)A, \tag{18}$$

where the Bond number $B = \frac{\rho g d^2}{2S}$. We need to consider the force balance in the tangential direction. Normally, at a free surface the stress is zero, but here the possibility of variation of surface tension along the surface requires the tangential viscous stress to equal the stress from surface tension. This leads to

$$\rho\nu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) = \frac{\partial S}{\partial x} = -\alpha S_0\frac{\partial(\delta T)}{\partial x} = -\alpha S_0\frac{(\Delta T)dH}{1 + dH}\cdot\frac{\partial(\theta + \eta)}{\partial x} \tag{19}$$

or

$$\left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial z^2}\right)w = -\frac{\alpha S_0 d^2(\Delta T)}{\rho\nu\lambda}\cdot\frac{dH}{1 + dH}(\theta_{xx} + \eta_{xx}) = -M(\theta_{xx} + \eta_{xx}). \tag{20}$$

In terms of W and Θ, then

$$(D^2 + a^2)W + Ma^2(\Theta + A) = 0. \tag{21}$$

Summarizing, the linear stability equations are

$$(D^2 - a^2)(D^2 - a^2 - p)W = \tau DZ, \tag{22}$$

$$(D^2 - a^2 - p)Z = -\tau DW, \tag{23}$$

$$(D^2 - a^2 - \sigma p)\Theta = -W, \tag{24}$$

with the boundary condition

$$W = DW = \Theta = Z = 0 \text{ on } z = 0 \text{ and}$$

$$DZ = 0,$$

$$W = pA,$$

$$D\Theta = \Gamma(\Theta + A),$$

$$Cr\left[-p + D^2 - 3a^2\right]DW = a^2(B + a^2)A,$$

$$(D^2 + a^2)W + Ma^2(\Theta + A) = 0 \text{ on } z = 1.$$

3. Hydrodynamic instabilities

In this section, we ignore the dissipation in the fluid and determine what the oscillatory modes of the linearized system are [5]. Dropping the viscous term in eq. (3) and trying out a solution $v = Ve^{i(k\cdot r - \omega t)}$ we have

$$-i\omega V_i = -ik_i P + 2\varepsilon_{ijk} V_j \Omega_k. \tag{25}$$

Incompressibility leads to

$$k_i V_i = 0, \tag{26}$$

$$k^2 P = 2\varepsilon_{ijk} k_i V_j \Omega_k. \tag{27}$$

and

$$V_i V_i = 0. \tag{28}$$

Choosing Ω in the z-direction

$$k^2 P = 2(k_x V_y - V_x k_y). \tag{29}$$

If the propagation direction is taken along the z-axis, then the transreversebility condition gives $V_z = 0$, and we have

$$V_x^2 + V_y^2 = 0. \tag{30}$$

The components of eq. (25) yield

$$-i\omega V_x = 2\Omega V_y, \tag{31}$$

$$-i\omega V_y = 2\Omega V_x, \tag{32}$$

and we have

$$\omega = 2\Omega. \tag{33}$$

Thus, in a virtually infinite mass of rotating fluid, there is a transverse wave of frequency 2Ω. In general, this wave will be damped out by viscosity.

An interesting variation of this wave motion is obtained when we consider the extension of the fluid in the z-direction to be 'd' on an average with fluctuations being denoted by η.

We first note that with the axis of rotation in the z-direction, the Coriolis force components are in the $x-y$ plane. If the pressure is generated due to gravity alone, then the z-component of the velocity will be virtually uneffected (Taylor-Proudman theorem). The horizontal acceleration will be given by

$$\frac{\partial u}{\partial t} = -\frac{\partial P}{\partial x} - 2\Omega v, \tag{34}$$

$$\frac{\partial v}{\partial t} = -\frac{\partial P}{\partial y} + 2\Omega u. \tag{35}$$

The right hand side depends on horizontal conditions only and hence the horizontal accelerations are functions of x and y alone. Consequently, the velocities u and v are functions of x and y and the incompressibility condition leads to

$$w = -z\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + f(x, y) \tag{36}$$

on $z = 0$, $w = 0$ and hence $f(x, y) = 0$. On the free surface $z = d + \eta\,(x, y)$, the pressure $P = -gz\rho = -\rho g d - g\eta\rho = P_0$ (constant) and the z-velocity w is given by

$$w = \frac{\partial \eta}{\partial t} + u\frac{\partial \eta}{\partial x} + v\frac{\partial \eta}{\partial z}, \tag{37}$$

$$w = -d\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) - \eta\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) \tag{38}$$

leading to

$$\frac{\partial \eta}{\partial t} + \frac{\partial}{\partial x}(u \cdot d) + \frac{\partial}{\partial y}(v \cdot d) + \frac{\partial}{\partial x}(u \cdot \eta) + \frac{\partial}{\partial y}(u \cdot \eta) = 0. \tag{39}$$

Linearizing

$$\frac{\partial \eta}{\partial t} + \frac{\partial}{\partial x}(u \cdot d) + \frac{\partial}{\partial y}(v \cdot d) = 0. \tag{40}$$

Combining eqs. (34) and (35) with $P = \rho g z + \rho g \eta(x, y)$

$$\frac{\partial u}{\partial t} = -g\frac{\partial \eta}{\partial x} - 2\Omega v, \tag{41}$$

$$\frac{\partial v}{\partial t} = -g\frac{\partial \eta}{\partial y} + 2\Omega u. \tag{42}$$

From eqs. (41) and (42)

$$\left(\frac{\partial^2}{\partial t^2} + 4\Omega^2\right)u = -g\frac{\partial^2 \eta}{\partial x \partial t} - 2\Omega g\frac{\partial \eta}{\partial y}, \tag{43}$$

$$\left(\frac{\partial^2}{\partial t^2} + 4\Omega^2\right)v = -g\frac{\partial^2 \eta}{\partial y \partial t} - 2\Omega g\frac{\partial \eta}{\partial x}. \tag{44}$$

Using eq. (40)

$$\left(\frac{\partial^2}{\partial t^2} + 4\Omega^2\right)\frac{\partial \eta}{\partial t} = gd\frac{\partial}{\partial t}\left(\frac{\partial^2 \eta}{\partial x^2} + \frac{\partial^2 \eta}{\partial y^2}\right) \tag{45}$$

If we consider the plane wave solution $\eta = Ae^{i(k_1 x + k_2 y) - i\omega t}$, then $\omega^2 = 4\Omega^2 + gdk^2$, where $k^2 = k_1^2 + k_2^2$. For $\Omega = 0$, we get the usual shallow water gravity wave of velocity $(gd)^2$. For large Ω, one gets a wave with frequency 2Ω, same as ω one had for the inertial wave discussed above.

We now bring in another variation. The geometry will be that of a channel, with the extension in the y-direction being $2L$, while the extension is still infinite in the x-direction.

We now try solution of the form

$$\eta = f(y)\, e^{i(kx - \omega t)}. \tag{46}$$

From eq. (45),

$$\frac{d^2 f}{dy^2} - k^2 f = \frac{4\Omega^2 - \omega^2}{gd} f. \tag{47}$$

The y– velocity has to vanish at $y = \pm L$, which implies

$$2\Omega\frac{\partial \eta}{\partial x} = \frac{\partial^2 \eta}{\partial y \partial t} \tag{48}$$

at $y = \pm L$ and hence

$$\frac{df}{dy} = -2\frac{k\Omega}{\omega} f \tag{49}$$

at $y = \pm L$. The solution of eq (47) is

$$f = C_1 Sin\left[\frac{\omega^2-(gdk^2+4\Omega^2)}{gd}\right]^{1/2} y + C_2 Cos\left[\frac{\omega^2-(gdk^2+4\Omega^2)}{gd}\right]^{1/2} y \quad (50)$$

and imposing boundary condition leads to

$$\omega^2-(gdk^2+4\Omega^2)+\frac{4k^2\Omega^2}{\omega^2}gd=0 \quad (51)$$

or

$$\frac{\omega^2}{gd}-\frac{gdk^2+4\Omega^2}{gd}=\frac{n^2\pi^2}{L^2}. \quad (52)$$

Thus, the possibilities are

$$\omega = 2\Omega,$$

$$= (gd)^{1/2}k \text{ and}$$

$$= \sqrt{\frac{n^2\pi^2}{L^2}gd+gdk^2+4\Omega^2}\,. \quad (53)$$

The modes for finite L are known as Poincare modes and mode with w directly proportional to k is called the Kelvin mode. These waves will in general be damped due to viscosity. If the energy provided by heating from below in our scenario of Section 1 for Marangoni convection can overcome the viscous dissipation at a sufficiently small Marangoni numbers, then the onset of convection can be in the form of a limit cycle.

4. Outlook

In this short section, we discuss what sort of results are known for the pure Marangoni convection and what could be useful directions for future study. For the situation without rotation, in the case of infinite aspect ratio, the critical Marangoni number and the corresponding wavenumber can be calculated exactly [3, 4]. The convection sets in as long wavelength rolls under certain conditions. It is only very recently that these long wavelength rolls have been experimentally observed [6]. The onset of convection in the infinite aspect ratio, nonrotating case, is always seen to be oscillatory, however, as yet no formal demonstration of the principle of exchange of stabilities exists. The effect of finite aspect ratio in the case, when the bottom plate is heated to produce convection and the sidewalls are insulting, has recently been found [7]. The critical Marangoni number is lower than that for the infinite aspect ratio situation. In a finite box, Marangoni convection can be induced by keeping a temperature difference between the sidewalls with the bottom plate insulating. In this case, under certain conditions oscillatory convection is indeed seen [8]. The underlying wave motion is the capillary wave in a fluid layer. Thus, we see that an oscillatory mode of the system can indeed be released as a limit cycle in the dissipative system in the presence of external forcing. In the case with rotation, there is an inertial wave which propagates, as well as the modified version of it in the presence of sidewalls.

In the case of Benard convection, the inertial wave mode is indeed released in the form of oscillatory convection for high rotation speeds. Whether this will happen in Marangoni convection is not yet settled. Thus, the question of overstability in the case of Marangoni convection with rotation is likely to be an interesting issue in the coming years.

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